

CONSTRUCTION OF LIE ALGEBRAS WITH SPECIAL G_2 -STRUCTURES

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ABSTRACT. We give a method to obtain new 7-dimensional Lie algebras endowed with closed and coclosed G_2 -structures starting from 6-dimensional Lie algebras with symplectic half-flat $SU(3)$ -structures and half-flat $SU(3)$ -structures, respectively. Finally, we describe all the 7-dimensional Lie algebras with a closed G_2 -structure that are obtained with this method from the 6-dimensional solvable Lie algebras admitting a symplectic half-flat $SU(3)$ -structure.

INTRODUCTION

An $SU(n)$ -structure on a Lie algebra \mathfrak{h} of dimension $2n$, consists in a triple (g, J, Ψ) such that (g, J) is an almost Hermitian structure on \mathfrak{h} , and $\Psi = \psi_+ + i\psi_-$ is a complex volume $(n, 0)$ -form, satisfying

$$(-1)^{n(n-1)/2} \left(\frac{i}{2}\right)^n \Psi \wedge \bar{\Psi} = \frac{1}{n!} \omega^n,$$

with $\bar{\Psi}$ the complex form obtained by conjugation of Ψ , and ω the Kähler form associated to (g, J) . In what follows we will consider $SU(3)$ -structures on 6-dimensional Lie algebras.

The existence of an $SU(3)$ -structure on a Lie algebra \mathfrak{h} can also be described by the presence of a pair of forms, namely, $(\omega, \psi_+) \in \Lambda^2 \mathfrak{h}^* \times \Lambda^3 \mathfrak{h}^*$ such that describe a metric as

$$g(X, Y) \omega^3 = -3 \iota_X \omega \wedge \iota_Y (\psi_+) \wedge \psi_+,$$

with $X, Y \in \mathfrak{h}$ and ι_X denoting the contraction by X . We can also recover its compatible almost complex structure as it is described in [7]

$$(J_{\psi_+}^* \alpha)(X) \omega^3 = \alpha \wedge \iota_X \psi_+ \wedge \psi_+,$$

or, equivalently,

$$\alpha(JX) = -J^* \alpha(X),$$

for any 1-form α on \mathfrak{h}^* .

Also, if (g, J, Ψ) is an $SU(3)$ -structure on a Lie algebra \mathfrak{h} we may choose an orthonormal frame $\{e_1, \dots, e_6\}$ such that the almost complex structure J is $J^* e^1 = e^2$, $J^* e^3 = e^4$ and $J^* e^5 = e^6$ with $\{e^1, \dots, e^6\}$ an orthonormal basis dual to

$\{e_1, \dots, e_6\}$. Therefore, the Kähler form ω and the complex volume form Ψ can be written as

$$(1) \quad \omega = e^{12} + e^{34} + e^{56}, \quad \Psi = (e^1 + i e^2) \wedge (e^3 + i e^4) \wedge (e^5 + i e^6),$$

where, with the usual notation of the related literature, we write e^{ij} for the wedge product $e^i \wedge e^j$, $e^{ijk} = e^i \wedge e^j \wedge e^k$, and so on. Thus,

$$\psi_+ = e^{135} - e^{146} - e^{236} - e^{245}, \quad \text{and} \quad \psi_- = -e^{246} + e^{235} + e^{145} + e^{136}.$$

In [12], Gray and Hervella prove that there exist sixteen different classes of almost Hermitian structures attending to the behavior of the covariant derivative of its Kähler form. Equivalently, the different classes of $SU(n)$ -structures can be defined in terms of the forms ω, ψ_+ and ψ_- . In particular we are interested in two classes of $SU(3)$ -structures which were defined respectively in [5] and [13] as follows:

- (g, J, Ψ) is a *half-flat* $SU(3)$ -structure iff $d\omega^2 = d\psi_+ = 0$;
- (g, J, Ψ) is a *symplectic half-flat* $SU(3)$ -structure iff $d\omega = d\psi_+ = 0$.

A classification of half-flat $SU(3)$ -structures on nilpotent Lie algebras is done in [3]. In [10] a similar work for indecomposable solvable Lie algebras has been established. The existence of symplectic half-flat $SU(3)$ -structures on nilpotent Lie algebras is studied in [6] and the complete study of these structures on solvable Lie algebras is obtained in [9].

A G_2 -structure on a 7-dimensional Lie algebra \mathfrak{g} is defined by a 3-form φ (called the fundamental form) on \mathfrak{g} such that

$$g_\varphi(X, Y) \text{ vol} = \frac{1}{6} \iota_X \varphi \wedge \iota_Y \varphi \wedge \varphi,$$

defines a Riemannian metric with $X, Y \in \mathfrak{g}$ and vol denoting the volume form. With respect to some orthonormal basis of 1-forms $\{e^1, \dots, e^7\}$ on \mathfrak{g} the fundamental form can be written as

$$(2) \quad \varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}.$$

It can also be defined the 4-form $*\varphi$, where $*$ denotes the Hodge star operator associated to g_φ . Therefore, respect to the basis $\{e^1, \dots, e^7\}$ of 1-forms of \mathfrak{g} in which the fundamental form is described by (2) the 4-form can be described as

$$\varphi = e^{1234} + e^{1256} + e^{1234} - e^{2467} + e^{2357} + e^{1457} + e^{1367}.$$

In [8], Fernández and Gray prove that there exist sixteen different classes of G_2 -structures attending to the behavior of the covariant derivative of its fundamental form. In particular we will be interested in two different classes of G_2 -structures which are described as follows:

- φ is an *almost parallel* or *closed* G_2 -structure iff $d\varphi = 0$;
- φ is a *semiparallel* or *coclosed* G_2 -structure iff $d*\varphi = 0$.

A classification of closed G_2 -structures on nilpotent Lie algebras has been recently obtained in [4].

$SU(3)$ -structures and G_2 -structures are closely related. In fact, if (N^6, ω, ψ_+) is a 6-dimensional manifold endowed with an $SU(3)$ -structure then the 3-form

$$(3) \quad \varphi = \omega \wedge dt + \psi_+,$$

defines a G_2 -structure on the 7-dimensional manifold $M^7 = N^6 \times S^1$ where t denotes the coordinate in S^1 .

Concerning the relation between special $SU(3)$ -structures and special G_2 -structures, if the $SU(3)$ -structure (ω, ψ_+) on N^6 is symplectic half-flat clearly the G_2 -structure defined by (3) constitutes a closed G_2 -structure on M^7 . Equivalently, if the $SU(3)$ manifold (N^6, ω, ψ_+) is half-flat the 3-form

$$(4) \quad \varphi = \omega \wedge dt - \psi_-,$$

is such that

$$*\varphi = \frac{1}{2}\omega \wedge \omega + \psi_+ \wedge dt,$$

and therefore defines a coclosed G_2 -structure on the 7-dimensional manifold $M^7 = N^6 \times S^1$ where t is the coordinate on S^1 .

Regarding the converse, Cabrera in [2] shows that if N^6 is an orientable hypersurface of a G_2 manifold (M^7, φ) then the pair of forms on N^6 defined by,

$$\omega = \iota_U \varphi \quad \text{and} \quad \psi_+ = \pi^* \varphi,$$

with U the unitary vector field of M^7 normal to N^6 and π the projection of M^7 onto N^6 , describe an $SU(3)$ -structure on N^6 .

If we focus our attention on Lie algebras it is clear that the presence of a symplectic half-flat structure namely (ω, ψ_+) on a 6-dimensional Lie algebra \mathfrak{h} , defines a closed G_2 -structure on $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ defined as in (3). Equivalently, if the $SU(3)$ -structure (ω, ψ_+) on \mathfrak{h} is half-flat the G_2 -structure defined on $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ by (4) is coclosed. These constructions of 7-dimensional Lie algebras endowed with closed and coclosed G_2 -structures as the direct sum of 6-dimensional Lie algebras with symplectic half-flat $SU(3)$ -structures or half-flat ones plus a 1-dimensional abelian Lie algebra are well known. In the present work we generalize this construction. This fact allows to obtain new examples of 7-dimensional Lie algebras endowed with closed and coclosed G_2 -structures. Thus, provided the existence of a lattice we can construct new compact solvmanifolds endowed with special G_2 -structures.

In Proposition 1.1 we describe how to obtain a 7-dimensional Lie algebra of the form

$$(5) \quad \mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}$$

endowed with a closed G_2 -structure from a 6-dimensional Lie algebra \mathfrak{h} with a symplectic half-flat $SU(3)$ -structure, where D denotes a derivation of \mathfrak{h} , what constitutes a generalization of some of the results obtained in [11]. Then, in Example 1.3 we give a concrete example of this construction. Concerning the

converse, in Proposition 1.4 we show how a closed G_2 -structure on a 7-dimensional Lie algebra of the form (5), with D a particular type of derivation, describes a symplectic half-flat $SU(3)$ -structure on the 6-dimensional subalgebra \mathfrak{h} . An example of this construction is described in Example 1.5.

Section 2 is devoted to an equivalent study but considering coclosed G_2 -structures and half-flat $SU(3)$ -structures. In particular, in Proposition 2.1 we describe how to obtain 7-dimensional Lie algebras endowed with a coclosed G_2 -structure from 6-dimensional Lie algebras with half-flat $SU(3)$ -structures. A concrete example of this construction is given in Example 2.4. Regarding the converse of Proposition 2.1 we show in Proposition 2.5 how a coclosed G_2 -structure on a 7-dimensional Lie algebra \mathfrak{g} of the form (5), with D a particular type of derivation, describes a half-flat $SU(3)$ -structure on the subalgebra \mathfrak{h} . Example 2.6 shows the use of Proposition 2.5.

1. LIE ALGEBRAS WITH A CLOSED G_2 -STRUCTURE

We show that if a 6-dimensional symplectic half-flat Lie algebra is endowed with a particular type of derivation, then one can construct a Lie algebra with a closed G_2 form. If \mathfrak{h} is a 6-dimensional Lie algebra, and D a derivation of \mathfrak{h} , the vector space

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi$$

is a Lie algebra with the Lie bracket given by

$$[U, V] = [U, V]_{|\mathfrak{h}}, \quad [\xi, U] = D(U),$$

for any $U, V \in \mathfrak{h}$.

We recall that a closed G_2 form on a real Lie algebra \mathfrak{g} of dimension 7 is a closed 3-form φ on \mathfrak{g} such that can be written as in (2) with respect to some basis $\{e^1, \dots, e^7\}$ of the dual space of \mathfrak{g} .

Let (ω, ψ_+) be a symplectic half-flat structure on \mathfrak{h} . Thus, it defines an almost complex structure J , and as it is mention on [1] this allows to obtain a real representation of the complex matrices as

$$\rho : \mathfrak{gl}(3, \mathbb{C}) \longrightarrow \mathfrak{gl}(6, \mathbb{R}).$$

Then, if $A \in \mathfrak{gl}(3, \mathbb{C})$, $\rho(A)$ is the matrix $(B_{ij})_{i,j=1}^3$ with

$$B_{ij} = \begin{pmatrix} \operatorname{Re} A_{ij} & \operatorname{Im} A_{ij} \\ -\operatorname{Im} A_{ij} & \operatorname{Re} A_{ij} \end{pmatrix},$$

where A_{ij} is the (i, j) component of A .

In particular, the real representation of $\mathfrak{sl}(3, \mathbb{C})$ (complex matrices without trace)

is given by

$$(6) \quad \mathfrak{sl}(3, \mathbb{C}) = \left\{ \left(\begin{array}{cc|cc|cc} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ -a_{1,2} & a_{1,1} & -a_{1,4} & a_{1,3} & -a_{1,6} & a_{1,5} \\ \hline a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \\ -a_{3,2} & a_{3,1} & -a_{3,4} & a_{3,3} & -a_{3,6} & a_{3,5} \\ \hline a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & -a_{1,1} - a_{3,3} & -a_{1,2} - a_{3,4} \\ -a_{5,2} & a_{5,1} & -a_{5,4} & a_{5,3} & a_{1,2} + a_{3,4} & -a_{1,1} - a_{3,3} \end{array} \right), \text{ with } a_{i,j} \in \mathbb{R} \right\}.$$

On the other hand, the $SU(3)$ -structure on \mathfrak{h} guarantees the existence of certain basis, namely $\{e^1, \dots, e^6\}$ of \mathfrak{h}^* , in which ω , ψ_+ and ψ_- have the canonical expression

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56}, \\ \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}, \\ \psi_- &= e^{136} + e^{145} + e^{235} - e^{246}. \end{aligned}$$

Proposition 1.1. *Let $(\mathfrak{h}, \omega, \psi_+)$ be a symplectic half-flat Lie algebra, and let D be a derivation of \mathfrak{h} such that D is the real representation of $A \in \mathfrak{sl}(3, \mathbb{C})$, with respect to a basis $\{e_1, \dots, e_6\}$ of \mathfrak{h} such that ω , ψ_+ and ψ_- have the canonical expression. Then, the Lie algebra*

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi,$$

has a closed G_2 form.

Proof. By (6) we have

$$D = \left(\begin{array}{cc|cc|cc} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ -a_{1,2} & a_{1,1} & -a_{1,4} & a_{1,3} & -a_{1,6} & a_{1,5} \\ \hline a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \\ -a_{3,2} & a_{3,1} & -a_{3,4} & a_{3,3} & -a_{3,6} & a_{3,5} \\ \hline a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & -a_{1,1} - a_{3,3} & -a_{1,2} - a_{3,4} \\ -a_{5,2} & a_{5,1} & -a_{5,4} & a_{5,3} & a_{1,2} + a_{3,4} & -a_{1,1} - a_{3,3} \end{array} \right),$$

with respect to the basis $\{e_1, \dots, e_6\}$ of \mathfrak{h} such that the $SU(3)$ -structure (ω, ψ_+) has the canonical expression.

Consider on \mathfrak{g} , the G_2 form

$$(7) \quad \varphi = \omega \wedge \eta + \psi_+,$$

with η the 1-form such that $\eta(X) = 0$ for all $X \in \mathfrak{h}$ and $\eta(\xi) = 1$.

For every $U, V, W, T \in \mathfrak{h}$

$$d\varphi(U, V, W, T) = d\psi_+(U, V, W, T),$$

which vanishes since ψ_+ is closed.

Hence, consider

$$\begin{aligned} d\varphi(U, V, W, \xi) &= -\varphi([U, V], W, \xi) + \varphi([U, W], V, \xi) - \varphi([U, \xi], V, W) \\ &\quad - \varphi([V, W], U, \xi) + \varphi([V, \xi], U, W) - \varphi([W, \xi], U, V), \end{aligned}$$

which by definition of φ is exactly

$$\begin{aligned} & -\omega([U, V], W) + \omega([U, W], V) - \omega([V, W], U) - \psi_+([U, \xi], V, W) \\ & + \psi_+([V, \xi], U, W) - \psi_+([W, \xi], U, V) = d\omega(U, V, W) + \psi_+(D(U), V, W) \\ & + \psi_+(U, D(V), W) + \psi_+(U, V, D(W)). \end{aligned}$$

Therefore, since ω is closed

$$d\varphi(U, V, W, \xi) = \psi_+(D(U), V, W) + \psi_+(U, D(V), W) + \psi_+(U, V, D(W)).$$

Taking into account the expression of D , and ψ_+ in terms of the basis $\{e_1, \dots, e_6\}$ an easy computation shows that

$$\psi_+(D(e_i), e_j, e_k) + \psi_+(e_i, D(e_j), e_k) + \psi_+(e_i, e_j, D(e_k)) = 0,$$

for every triple (e_i, e_j, e_k) of elements of the basis of \mathfrak{h} . Thus, the G_2 form φ defined in (7) is closed in \mathfrak{g} . \square

The previous proposition describes a method to construct new 7-dimensional Lie algebras with a closed G_2 -structure.

Remark 1.2. *Note that the trace of D , the real representation of certain $A \in \mathfrak{sl}(3, \mathbb{C})$ vanishes. Therefore, the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7$ will be unimodular if and only if \mathfrak{h} is so.*

Example 1.3. *Next, we show a new example of compact solvmanifold with closed G_2 form. Let \mathfrak{h} be the 6-dimensional abelian Lie algebra defined by the structure equations*

$$\mathfrak{h} = (0, 0, 0, 0, 0, 0),$$

The almost Hermitian structure (g, J) on \mathfrak{h} given by

$$g = \sum_{i=1}^6 e^i \otimes e^i, \quad Je_1 = e_2, \quad Je_3 = e_4, \quad Je_5 = e_6$$

is such that its Kähler form is

$$\omega = e^{12} + e^{34} + e^{56}.$$

Thus, (g, J) together with the complex volume form $\Psi = \psi_+ + i\psi_-$, where

$$\begin{aligned} \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}, \\ \psi_- &= e^{136} + e^{145} + e^{235} - e^{246}, \end{aligned}$$

define an $SU(3)$ -structure on \mathfrak{h} . Clearly, $d\omega^2 = d\psi_+ = 0$, so $(g, J, \Psi = \psi_+ + i\psi_-)$ is a half-flat $SU(3)$ -structure on \mathfrak{h} .

Consider now the derivation D of \mathfrak{h} given by

$$\begin{pmatrix} 1 & & \\ & 1 & \\ \hline & -1 & \\ & & -1 \\ \hline & & & 0 \\ & & & & 0 \end{pmatrix} \in \mathfrak{sl}(3, \mathbb{C}),$$

that is,

$$D(e_1) = e_1, \quad D(e_2) = e_2, \quad D(e_3) = -e_3, \quad D(e_4) = -e_4,$$

Take the Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

whose structure equations are

$$\mathfrak{g} = (e^{17}, e^{27}, -e^{37}, -e^{47}, 0, 0, 0).$$

Then, the 3-form φ given by

$$\varphi = e^{127} + e^{347} + e^{567} + e^{136} + e^{145} + e^{235} - e^{246}$$

is a closed G_2 form on \mathfrak{g} .

Lets denote by G the simply connected and completely solvable Lie group consisting on matrices of the form.

$$a = \left(\begin{array}{cc|cc|cc} e^{x_7} & & & & & x_1 \\ & e^{x_7} & & & & x_2 \\ \hline & & e^{-x_7} & & & x_3 \\ & & & e^{-x_7} & & x_4 \\ \hline & & & & 1 & x_5 \\ & & & & & 1 \\ \hline & & & & & 1 \\ & & & & 1 & x_7 \\ & & & & & 1 \end{array} \right),$$

with $x_i \in \mathbb{R}$, for $i = 1, \dots, 7$. Then a global system of coordinates $\{x_i\}$ for G is defined by $x_i(a) = x_i$. An standard calculation shows that a basis for the left invariant 1-forms on G can be described by

$$\begin{aligned} e^1 &= e^{-x_7} dx_1, & e^2 &= e^{-x_7} dx_2, & e^3 &= e^{x_7} dx_3, & e^4 &= e^{x_7} dx_4, \\ e^5 &= dx_5, & e^6 &= dx_6, & \text{and} & & e^7 &= dx_7. \end{aligned}$$

Therefore \mathfrak{g} is exactly the Lie algebra of G . Notice that $G = \mathbb{R} \ltimes_{\phi} \mathbb{R}^6$, where \mathbb{R} acts on \mathbb{R}^6 via ϕ_t described by

$$\phi_t = \left(\begin{array}{cc|cc|cc} e^t & & & & & \\ & e^t & & & & \\ \hline & & e^{-t} & & & \\ & & & e^{-t} & & \\ \hline & & & & 1 & \\ & & & & & 1 \end{array} \right).$$

Thus the operation on the group G is given by

$$a \cdot b = (b_1 e^{a_7} + a_1, b_2 e^{a_7} + a_2, b_3 e^{-a_7} + a_3, b_4 e^{-a_7} + a_4, b_5 + a_5, b_6 + a_6, b_7 + a_7),$$

where $a = (a_1, \dots, a_7)$ and $b = (b_1, \dots, b_7)$.

To construct the lattice Γ of G it is enough to find some real number t_0 such that ϕ_{t_0} is conjugated to an element $A \in SL(6, \mathbb{Z})$. If Γ_0 denotes a lattice of \mathbb{R}^6 invariant under ϕ_{t_0} , take

$$\Gamma = (t_0 \mathbb{Z}) \ltimes_{\phi} \Gamma_0.$$

Consider the matrix

$$A = \left(\begin{array}{cc|cc|cc} 2 & 1 & & & & \\ 1 & 1 & & & & \\ \hline & & 2 & 1 & & \\ & & 1 & 1 & & \\ \hline & & & & 1 & \\ & & & & & 1 \end{array} \right),$$

with double eigenvalues $\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$. Taking $t_0 = \text{Ln}(\frac{3+\sqrt{5}}{2})$ we have that $e^{t_0 D}$ and A are conjugated. In particular, take

$$P = \left(\begin{array}{cc|cc|cc} 1 & \frac{-1+\sqrt{5}}{2} & & & & \\ 1 & \frac{-1-\sqrt{5}}{2} & & & & \\ \hline & & 1 & \frac{-1+\sqrt{5}}{2} & & \\ & & 1 & \frac{-1-\sqrt{5}}{2} & & \\ \hline & & & & 1 & \\ & & & & & 1 \end{array} \right),$$

it is easy to check that $PA = \phi_{t_0} P$. So, the lattice defined by

$$\Gamma_0 = P \mathbb{Z} \langle e_1, \dots, e_6 \rangle$$

is invariant under the group $t_0 \mathbb{Z}$. Thus

$$\Gamma = (t_0 \mathbb{Z}) \ltimes_{\phi} \Gamma_0$$

is a lattice of G . Then, the compact solvmanifold $S = \Gamma \backslash G$ admits a closed G_2 -structure.

Regarding the converse of Proposition 1.1 we have the following result

Proposition 1.4. *Let φ be the closed G_2 form*

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

on the 7-dimensional Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

where D is a derivation of $\mathfrak{h} = \mathfrak{g}/\langle e_7 \rangle$ such that it is given with respect to the basis $\{e_1, \dots, e_6\}$ by a matrix of the form (6). Then, the 6-dimensional Lie algebra \mathfrak{h} has a symplectic half-flat structure.

Proof. Consider the pair of forms (ω, ψ_+) on \mathfrak{h} defined by

$$\omega = \iota_{e_7} \varphi \quad \text{and} \quad \psi_+ = \pi^* \varphi,$$

with π the projection of \mathfrak{g} onto \mathfrak{h} . Thus (ω, ψ_+) define an $SU(3)$ -structure on \mathfrak{h} , which in terms of the basis $\{e_1, \dots, e_6\}$ of \mathfrak{h} has the canonical expression, that is

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56}, \\ \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}. \end{aligned}$$

Then, for every $U, V, W, T \in \mathfrak{h}$

$$d\psi_+(U, V, W, T) = d\varphi(U, V, W, T),$$

which vanishes since φ is closed, and therefore ψ_+ is closed.

Taking into account the expression of D , and ψ_+ in terms of the basis $\{e_1, \dots, e_6\}$ an easy computation shows that

$$\psi_+(D(e_i), e_j, e_k) + \psi_+(e_i, D(e_j), e_k) + \psi_+(e_i, e_j, D(e_k)) = 0,$$

for every triple (e_i, e_j, e_k) of elements of the basis of \mathfrak{h} . Consider now $d\varphi(U, V, W, e_7)$ which is exactly

$$\begin{aligned} d\varphi(U, V, W, e_7) = & d\omega(U, V, W) + \psi_+(D(U), V, W) \\ & + \psi_+(U, D(V), W) + \psi_+(U, V, D(W)). \end{aligned}$$

Therefore, since D is the real representation of certain $A \in \mathfrak{sl}(3, \mathbb{C})$

$$d\omega(U, V, W) = d\varphi(U, V, W, e_7),$$

which vanishes since φ is closed. Thus the $SU(3)$ -structure on \mathfrak{h} given by the pair (ω, ψ_+) is symplectic half-flat. \square

Example 1.5. Let \mathfrak{g} be the 7-dimensional nilpotent Lie algebra described by the structure equations

$$\mathfrak{g} = (0, 0, e^{17}, e^{15} + e^{27}, 0, e^{13}, 0).$$

Then, the G_2 form given by

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

is closed.

Notice that \mathfrak{g} is of the form

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

where the 6-dimensional Lie algebra $\mathfrak{h} = \mathfrak{g}/\langle e_7 \rangle$ is described by the structure equations

$$\mathfrak{h} = (0, 0, 0, e^{15}, 0, e^{13}).$$

The derivation D of \mathfrak{h} , is given with respect to the basis $\{e_1, \dots, e_6\}$ by the matrix

$$D = \begin{pmatrix} & & 1 & & & \\ & & & 1 & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix}.$$

Therefore D is the real representation of certain $A \in \mathfrak{sl}(3, \mathbb{C})$. Then, the $SU(3)$ -structure defined by

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56}, \\ \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}. \end{aligned}$$

is symplectic half-flat.

As a consequence of Proposition 1.1 and 1.4 we have the following result:

Theorem 1.6. *Let \mathfrak{h} be a 6-dimensional Lie algebra and let \mathfrak{g} be a 7-dimensional Lie algebra satisfying*

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

with D a derivation of \mathfrak{h} given by (6) in terms of a basis $\{e_1, \dots, e_6\}$ of \mathfrak{h} . Then the following two conditions are equivalent:

- (1) *The $SU(3)$ -structure on \mathfrak{h} given by*

$$\begin{aligned}\omega &= e^{12} + e^{34} + e^{56}, \\ \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}.\end{aligned}$$

is symplectic half-flat.

- (2) *The G_2 -structure on \mathfrak{g} given by*

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

is closed.

2. LIE ALGEBRAS WITH A COCLOSED G_2 -STRUCTURE

In this section we show that if a 6-dimensional half-flat Lie algebra is endowed with a particular type of derivation, then a Lie algebra with a coclosed G_2 -structure can be constructed.

We recall that a coclosed G_2 -structure on a real Lie algebra \mathfrak{g} of dimension 7 consists on the presence of a G_2 form which is coclosed. In order to obtain an expression adapted to our purposes, in this section we characterize a G_2 form on \mathfrak{g} as a 3-form that can be written as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{246} - e^{235} - e^{145} - e^{136},$$

with respect to some basis $\{e^1, \dots, e^7\}$ of the dual space of \mathfrak{g} .

Proposition 2.1. *Let $(\mathfrak{h}, \omega, \psi_+)$ be a half-flat Lie algebra, and let D be a derivation of \mathfrak{h} such that $D \in \mathfrak{sp}(6, \mathbb{R})$, with respect to a basis $\{e_1, \dots, e_6\}$ of \mathfrak{h} satisfying that ω , ψ_+ and ψ_- have the canonical expression. Then, the Lie algebra*

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi,$$

has a coclosed G_2 form.

Proof. Since $D \in \mathfrak{sp}(6, \mathbb{R})$ with respect to the basis $\{e_1, \dots, e_6\}$, we can write

$$(8) \quad D = \left(\begin{array}{cc|cc|cc} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ a_{2,1} & -a_{1,1} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} \\ \hline -a_{2,4} & a_{1,4} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \\ a_{2,3} & -a_{1,3} & a_{4,3} & -a_{3,3} & a_{4,5} & a_{4,6} \\ \hline -a_{2,6} & a_{1,6} & -a_{4,6} & a_{3,6} & a_{5,5} & a_{5,6} \\ a_{2,5} & -a_{1,5} & a_{4,5} & -a_{3,5} & a_{6,5} & -a_{5,5} \end{array} \right).$$

Consider on \mathfrak{g} , the G_2 form

$$(9) \quad \varphi = \omega \wedge \eta - \psi_-,$$

thus

$$*\varphi = \frac{1}{2}\omega \wedge \omega + \psi_+ \wedge \eta,$$

where η is the 1-form satisfying that $\eta(X) = 0$ for all $X \in \mathfrak{h}$ and $\eta(\xi) = 1$. For every $U, V, W, T, R \in \mathfrak{h}$,

$$d*\varphi(U, V, W, T, R) = d\omega \wedge \omega(U, V, W, T, R),$$

which vanishes since $\omega \wedge \omega$ is closed.

Hence, consider

$$\begin{aligned} d*\varphi(U, V, W, T, \xi) = & -*\varphi([U, V], W, T, \xi) + *\varphi([U, W], V, T, \xi) - *\varphi([U, T], V, W, \xi) \\ & + *\varphi([U, \xi], V, W, T) - *\varphi([V, W], U, T, \xi) + *\varphi([V, T], U, W, \xi) \\ & - *\varphi([V, \xi], U, W, T) - *\varphi([W, T], U, V, \xi) + *\varphi([W, \xi], U, V, T) \\ & - *\varphi([T, \xi], U, V, W), \end{aligned}$$

which by the definition of $*\varphi$ is exactly

$$\begin{aligned} & -\psi_+([U, V], W, T) + \psi_+([U, W], V, T) - \psi_+([U, T], V, W) - \psi_+([V, W], U, T) \\ & + \psi_+([V, T], U, W) - \psi_+([W, T], U, V) + \frac{1}{2}\omega \wedge \omega([U, \xi], V, W, T) \\ & - \frac{1}{2}\omega \wedge \omega([V, \xi], U, W, T) + \frac{1}{2}\omega \wedge \omega([W, \xi], U, V, T) - \frac{1}{2}\omega \wedge \omega([T, \xi], U, V, W) \\ & = d\psi_+(U, V, W, T) + \frac{1}{2}\omega \wedge \omega(D(U), V, W, T) + \frac{1}{2}\omega \wedge \omega(U, D(V), W, T) \\ & + \frac{1}{2}\omega \wedge \omega(U, V, D(W), T) + \frac{1}{2}\omega \wedge \omega(U, V, W, D(T)). \end{aligned}$$

Therefore since ψ_+ is closed

$$\begin{aligned} d*\varphi(U, V, W, T, \xi) = & \frac{1}{2}\omega \wedge \omega(D(U), V, W, T) + \frac{1}{2}\omega \wedge \omega(U, D(V), W, T) \\ & + \frac{1}{2}\omega \wedge \omega(U, V, D(W), T) + \frac{1}{2}\omega \wedge \omega(U, V, W, D(T)). \end{aligned}$$

Using the expressions of D and ω with respect to the basis $\{e_1, \dots, e_6\}$, can be checked that

$$\begin{aligned} & \omega \wedge \omega(D(e_i), e_j, e_k, e_l) + \omega \wedge \omega(e_i, D(e_j), e_k, e_l) \\ & + \omega \wedge \omega(e_i, e_j, D(e_k), e_l) + \omega \wedge \omega(e_i, e_j, e_k, D(e_l)) = 0, \end{aligned}$$

for every quadruplet (e_i, e_j, e_k, e_l) of elements of the basis of \mathfrak{h} . Thus, the G_2 form φ defined in (9) is coclosed in \mathfrak{g} . \square

Previous proposition describes a method to construct 7-dimensional Lie algebras with a coclosed G_2 -structure.

Remark 2.2. *Note that the trace of $D \in \mathfrak{sp}(6, \mathbb{R})$ vanishes. Therefore, the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7$ will be unimodular if and only if \mathfrak{h} is so.*

Example 2.3. *Next, we show a new example of solvable Lie algebra with coclosed G_2 form. Let \mathfrak{h} be the 6-dimensional abelian Lie algebra defined by the structure equations*

$$\mathfrak{h} = (0, 0, 0, 0, 0, 0),$$

The almost Hermitian structure (g, J) on \mathfrak{h} given by

$$g = \sum_{i=1}^6 e^i \otimes e^i, \quad Je_1 = e_2, \quad Je_3 = e_4, \quad Je_5 = e_6$$

is such that its Kähler form is

$$\omega = e^{12} + e^{34} + e^{56}.$$

Thus, (g, J) together with the complex volume form $\Psi = \psi_+ + i\psi_-$, where

$$\begin{aligned} \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}, \\ \psi_- &= e^{136} + e^{145} + e^{235} - e^{246}, \end{aligned}$$

define an $SU(3)$ -structure on \mathfrak{h} . Clearly, $d\omega^2 = d\psi_+ = 0$, so $(g, J, \Psi = \psi_+ + i\psi_-)$ is a half-flat $SU(3)$ -structure on \mathfrak{h} .

Consider now the derivation D of \mathfrak{h} given by

$$\left(\begin{array}{c|c|c} 1 & & \\ \hline & -1 & \\ \hline & & 1 \\ & & \hline & & -1 \\ \hline & & & 1 \\ & & & \hline & & & -1 \end{array} \right) \in \mathfrak{sp}(6, \mathbb{R}),$$

that is,

$$\begin{aligned} D(e_1) &= e_1, & D(e_2) &= -e_2, & D(e_3) &= e_3, \\ D(e_4) &= -e_4, & D(e_5) &= e_5, & D(e_6) &= -e_6. \end{aligned}$$

Take the Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

whose structure equations are

$$\mathfrak{g} = (e^{17}, -e^{27}, e^{37}, -e^{47}, e^{57}, -e^{67}, 0).$$

Then, the 3-form φ given by

$$\varphi = e^{127} + e^{347} + e^{567} + e^{136} + e^{145} + e^{235} - e^{246}$$

is a closed G_2 form on \mathfrak{g} .

Example 2.4. *Let \mathfrak{h} be the 6-dimensional abelian Lie algebra described by the structure equations*

$$\mathfrak{h} = (0, 0, 0, 0, 0, 0).$$

The almost Hermitian structure given by

$$g = \sum_{i=1}^6 e^i \otimes e^i \quad \text{and} \quad Je_1 = e_2, \quad Je_3 = e_4, \quad Je_5 = e_6,$$

is such that its Kähler form is

$$\omega = e^{12} + e^{34} + e^{56}.$$

Thus, (g, J) together with the complex volume form $\Psi = \psi_+ + i\psi_-$ where

$$\begin{aligned}\psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}, \\ \psi_- &= e^{136} + e^{145} + e^{235} - e^{246},\end{aligned}$$

define an $SU(3)$ -structure on \mathfrak{h} . Concretely, since ω^2 and ψ_+ are closed it is a half-flat structure. Consider now the derivation D of \mathfrak{h} given by

$$D = \left(\begin{array}{c|c|c} 1 & & \\ \hline & -1 & \\ \hline & & 1 \\ & & -1 \\ \hline & & & 1 \\ & & & -1 \end{array} \right) \in \mathfrak{sp}(6, \mathbb{R}),$$

that is,

$$\begin{aligned}D(e_1) &= e_1, & D(e_2) &= -e_2, & D(e_3) &= e_3, \\ D(e_4) &= -e_4, & D(e_5) &= e_5 & \text{and} & D(e_6) = -e_6.\end{aligned}$$

Thus, the Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

which is described by the structure equations

$$\mathfrak{g} = (e^{17}, -e^{27}, e^{37}, -e^{47}, e^{57}, -e^{67}, 0),$$

is completely solvable and admits the coclosed G_2 form

$$\varphi = e^{127} + e^{347} + e^{567} + e^{136} + e^{145} + e^{235} - e^{246}.$$

Lets denote by G the simply connected and completely solvable Lie group consisting on matrices of the form.

$$a = \left(\begin{array}{c|c|c|c} e^{x_7} & & & x_1 \\ & e^{-x_7} & & x_2 \\ \hline & & e^{x_7} & x_3 \\ & & & e^{-x_7} \\ \hline & & & e^{x_7} \\ & & & e^{-x_7} \\ \hline & & & 1 & x_7 \\ & & & & 1 \end{array} \right),$$

with $x_i \in \mathbb{R}$, for $i = 1, \dots, 7$. Then a global system of coordinates $\{x_i\}$ for G is defined by $x_i(a) = x_i$. An standard calculation shows that a basis for the left invariant 1-forms on G can be described by

$$\begin{aligned}e^1 &= e^{-x_7} dx_1, & e^2 &= e^{x_7} dx_2, & e^3 &= e^{-x_7} dx_3, & e^4 &= e^{x_7} dx_4, \\ e^5 &= e^{-x_7} dx_5, & e^6 &= e^{-x_7} dx_6, & \text{and} & e^7 &= dx_7.\end{aligned}$$

Therefore \mathfrak{g} is exactly the Lie algebra of G . Notice that $G = \mathbb{R} \ltimes_{\phi} \mathbb{R}^6$, where \mathbb{R} acts on \mathbb{R}^6 via ϕ_t described by

$$\phi_t = \left(\begin{array}{c|c|c} e^t & & \\ \hline & e^{-t} & \\ \hline & & e^t \\ \hline & & & e^{-t} \end{array} \right).$$

Thus the operation on the group G is given by

$$a \cdot b = (b_1 e^{a_7} + a_1, b_2 e^{-a_7} + a_2, b_3 e^{a_7} + a_3, b_4 e^{-a_7} + a_4, b_5 e^{a_7} + a_5, b_6 e^{-a_7} + a_6, b_7 + a_7),$$

where $a = (a_1, \dots, a_7)$ and $b = (b_1, \dots, b_7)$.

To construct the lattice Γ of G it is enough to find some real number t_0 such that ϕ_{t_0} is conjugated to an element $A \in SL(6, \mathbb{Z})$. If Γ_0 denotes a lattice of \mathbb{R}^6 invariant under ϕ_{t_0} , take

$$\Gamma = (t_0 \mathbb{Z}) \ltimes_{\phi} \Gamma_0.$$

Consider the matrix

$$A = \left(\begin{array}{cc|cc|cc} 2 & 1 & & & & \\ 1 & 1 & & & & \\ \hline & & 2 & 1 & & \\ & & 1 & 1 & & \\ \hline & & & & 2 & 1 \\ & & & & 1 & 1 \end{array} \right),$$

with triple eigenvalues $\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$. Taking $t_0 = \ln(\frac{3+\sqrt{5}}{2})$ we have that $e^{t_0 D}$ and A are conjugated. In particular, take

$$P = \left(\begin{array}{cc|cc|cc} 1 & \frac{-1+\sqrt{5}}{2} & & & & \\ 1 & \frac{-1-\sqrt{5}}{2} & & & & \\ \hline & & 1 & \frac{-1+\sqrt{5}}{2} & & \\ & & 1 & \frac{-1-\sqrt{5}}{2} & & \\ \hline & & & & 1 & \frac{-1+\sqrt{5}}{2} \\ & & & & 1 & \frac{-1-\sqrt{5}}{2} \end{array} \right),$$

it is easy to check that $PA = \phi_{t_0} P$. So, the lattice defined by

$$\Gamma_0 = P \mathbb{Z} \langle e_1, \dots, e_6 \rangle$$

is invariant under the group $t_0 \mathbb{Z}$. Thus

$$\Gamma = (t_0 \mathbb{Z}) \ltimes_{\phi} \Gamma_0$$

is a lattice of G . Then, the compact solvmanifold $S = \Gamma \backslash G$ admits a coclosed G_2 -structure.

Considering the converse of Proposition 2.1 we obtain the next result.

Proposition 2.5. *Let φ be the coclosed G_2 form*

$$\varphi = e^{127} + e^{347} + e^{567} + e^{136} - e^{145} - e^{235} - e^{246},$$

on the 7-dimensional Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

where D is a derivation of $\mathfrak{h} = \mathfrak{g}/\langle e_7 \rangle$ such that it is given with respect to the basis $\{e_1, \dots, e_6\}$ by a matrix of the form (8). Then, the 6-dimensional Lie algebra \mathfrak{h} has a half-flat structure.

Proof. Consider the pair of forms (ω, ψ_+) on \mathfrak{h} defined by

$$\omega = \iota_{e_7}\varphi \quad \text{and} \quad \psi_+ = *\psi_-,$$

where

$$\psi_- = -\pi^*\varphi,$$

with π the projection of \mathfrak{g} onto \mathfrak{h} . Thus (ω, ψ_+) define an $SU(3)$ -structure on \mathfrak{h} , which in terms of the basis $\{e_1, \dots, e_6\}$ of \mathfrak{h} has the canonical expression, that is

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56}, \\ \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}. \end{aligned}$$

Therefore we have that

$$*\varphi = \frac{1}{2}\omega \wedge \omega + \psi_+ \wedge e^7,$$

and then, for every $U, V, W, T, R \in \mathfrak{h}$

$$d\omega \wedge \omega(U, V, W, T, R) = d*\varphi(U, V, W, T, R),$$

which vanishes since $*\varphi$ is closed. Thus, $\omega \wedge \omega$ is closed.

Consider now $d*\varphi(U, V, W, T, e_7)$ which by definition of $*\varphi$ is exactly

$$\begin{aligned} d*\varphi(U, V, W, T, e_7) &= \omega \wedge \omega(D(U), V, W, T) + \omega \wedge \omega(U, D(V), W, T) \\ &\quad + \omega \wedge \omega(U, V, D(W), T) + \omega \wedge \omega(U, V, W, D(T)) \\ &\quad + d\psi_+(U, V, W, T). \end{aligned}$$

Since $D \in \mathfrak{sp}(6, \mathbb{R})$

$$d\psi_+(U, V, W, T) = d*\varphi(U, V, W, T, e_7),$$

that vanishes because φ is coclosed. Therefore the $SU(3)$ -structure on \mathfrak{h} given by the pair (ω, ψ_+) is half-flat. \square

Example 2.6. *Let \mathfrak{g} be the 7-dimensional Lie algebra described by the structure equations*

$$\mathfrak{g} = (e^{35} + e^{46}, 0, e^{67}, e^{57}, e^{47}, e^{37}, 0),$$

then, the G_2 form given by

$$\varphi = e^{127} + e^{347} + e^{567} + e^{246} - e^{235} - e^{136} - e^{145},$$

is coclosed, that is, the 4-form

$$*\varphi = e^{1234} + e^{1256} + e^{3456} + e^{1357} - e^{1467} - e^{2367} - e^{2457},$$

is closed. Notice that \mathfrak{g} is of the form

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

where the 6-dimensional Lie algebra $\mathfrak{h} = \mathfrak{g}/\langle e_7 \rangle$ is described by the structure equations

$$\mathfrak{h} = (e^{35} + e^{46}, 0, 0, 0, 0, 0).$$

The derivation D of \mathfrak{h} , is given with respect to the basis $\{e_1, \dots, e_6\}$ by the matrix

$$D = \begin{pmatrix} & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix}.$$

Therefore $D \in \mathfrak{sp}(6, \mathbb{R})$. Then, the $SU(3)$ -structure defined by

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56}, \\ \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}. \end{aligned}$$

is half-flat.

As a consequence of Propositions 2.1 and 2.5 we have the following:

Theorem 2.7. *Let \mathfrak{h} be a 6-dimensional Lie algebra and let \mathfrak{g} be a 7-dimensional Lie algebra satisfying*

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

with D a derivation of \mathfrak{h} given by (8) in terms of a basis $\{e_1, \dots, e_6\}$ of \mathfrak{h} . Then the following two conditions are equivalent:

- (1) *The $SU(3)$ -structure on \mathfrak{h} given by*

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56}, \\ \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}. \end{aligned}$$

is half-flat.

- (2) *The G_2 -structure on \mathfrak{g} given by*

$$\varphi = e^{127} + e^{347} + e^{567} + e^{136} + e^{145} + e^{235} - e^{246},$$

is coclosed.

3. CONSTRUCTION OF LIE ALGEBRAS WITH CLOSED G_2 -STRUCTURES

Finally, using the results of Section 1 we describe all the 7-dimensional Lie algebras with closed G_2 -structures which are constructed as

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

where \mathfrak{h} denotes a 6-dimensional solvable Lie algebra with a symplectic half-flat structure and D is a derivation of \mathfrak{h} . From [9] the 6-dimensional Lie algebras with a symplectic half-flat $SU(3)$ -structure are:

$$\begin{aligned}
\mathfrak{a} &= (0, 0, 0, 0, 0, 0); \\
\mathfrak{e}(1, 1) \oplus \mathfrak{e}(1, 1) &= (0, 0, -e^{14}, -e^{13}, e^{25}, -e^{26}); \\
\mathfrak{g}_{5,1} \oplus \mathbb{R} &= (0, 0, 0, e^{15}, 0, e^{13}); \\
\mathfrak{g}_{5,7^{-1},-1,1} \oplus \mathbb{R} &= (-e^{15}, e^{25}, -e^{35}, e^{45}, 0, 0); \\
\mathfrak{g}_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R} &= (\alpha e^{15} + e^{35}, -\alpha e^{25} + e^{45}, -e^{15} + \alpha e^{35}, -e^{25} - \alpha e^{45}, 0, 0); \\
\mathfrak{g}_{6,N3} &= (0, e^{35}, 0, 2e^{15}, 0, e^{13}); \\
\mathfrak{g}_{6,38}^0 &= (2e^{36}, 0, -e^{26}, -e^{26} + e^{25}, -e^{23} - e^{24}, e^{23}); \\
\mathfrak{g}_{6,54}^{0,-1} &= (e^{16} + e^{45}, -e^{26}, -e^{36} + e^{25}, e^{46}, 0, 0); \\
\mathfrak{g}_{6,118}^{0,-1,-1} &= (-e^{15} + e^{36}, e^{46} + e^{25}, -e^{16} - e^{35}, -e^{45} - e^{26}, 0, 0); \\
A_{6,13}^{-\frac{2}{3},\frac{1}{3},-1} &= (-\frac{1}{4}e^{14} - e^{23}, \frac{1}{4}e^{24}, -e^{26}, -e^{26} + e^{25}, -e^{23} - e^{24}, e^{23}); \\
A_{6,54}^{2,1} &= (-\frac{1}{2}e^{15}, \frac{1}{2}e^{25} + e^{16}, -\frac{1}{2}e^{35}, \frac{1}{2}e^{45} + e^{36}, 0, -e^{56}); \\
A_{6,70}^{\alpha,\frac{\alpha}{2}}(\alpha \neq 0) &= (-\frac{1}{2}e^{15} + \frac{1}{\alpha}e^{35} + e^{26}, \frac{1}{2}e^{25} + \frac{1}{\alpha}e^{45}, -\frac{1}{\alpha}e^{15} - \frac{1}{2}e^{35} + e^{46}, \\
&\quad -\frac{1}{\alpha}e^{25} + \frac{1}{2}e^{45}, 0, e^{56}); \\
A_{6,71}^{-\frac{3}{2}} &= (-\frac{3}{4}e^{16}, \frac{3}{4}e^{26} + e^{35}, \frac{1}{4}e^{36} + e^{45}, -\frac{1}{4}e^{46} + e^{15}, \frac{1}{2}e^{56}, 0); \\
N_{6,3}^{0,-2,0,2} &= (-2 \cdot 3^{1/3}e^{16}, 2 \cdot 3^{-1/3}e^{26}, 3^{-1/3}e^{36} + 3^{2/3}e^{45}, 0, \\
&\quad -3^{-1/3}e^{34} - 3^{-1/3}e^{56}, 0).
\end{aligned}$$

The structure equations of the previously mentioned Lie algebras are given in terms of an adapted basis, that is, a basis such that the forms

$$\begin{aligned}
\omega &= e^{12} + e^{34} + e^{56}, \\
\psi_+ &= e^{135} - e^{146} - e^{236} - e^{245},
\end{aligned}$$

are closed and therefore describe a symplectic half-flat $SU(3)$ -structure.

Proposition 3.1. *The Lie algebras described in Table 1 of the Appendix admit the closed G_2 -structure given by the 3-form (2), that is*

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}.$$

Proof. For each one of the 6-dimensional solvable Lie algebras admitting a symplectic half-flat $SU(3)$ -structure, namely \mathfrak{h} we consider the Lie algebras

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

with D being the real representation of certain $A \in \mathfrak{sl}(3, \mathbb{C})$, that is, D is of the form

$$D = \left(\begin{array}{cc|cc|cc} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ -a_{1,2} & a_{1,1} & -a_{1,4} & a_{1,3} & -a_{1,6} & a_{1,5} \\ \hline a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \\ -a_{3,2} & a_{3,1} & -a_{3,4} & a_{3,3} & -a_{3,6} & a_{3,5} \\ \hline a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & -a_{1,1} - a_{3,3} & -a_{1,2} - a_{3,4} \\ -a_{5,2} & a_{5,1} & -a_{5,4} & a_{5,3} & a_{1,2} + a_{3,4} & -a_{1,1} - a_{3,3} \end{array} \right).$$

Then, we compute for which values of the parameters $a_{i,j}$, the matrix D represents a derivation of \mathfrak{h} and thus, from Proposition 1.1 the 3-form

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$$

defines a closed G_2 -structure on $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7$.

- $\mathfrak{h} = \mathfrak{a}$

The structure equations of

$$\mathfrak{g} = \mathfrak{a} \oplus_D \mathbb{R}e_7,$$

with D a derivation described as in (6) are

$$\begin{aligned} (10) \quad de^1 &= a_{1,1}e^{17} - a_{1,2}e^{27} + a_{3,1}e^{37} - a_{3,2}e^{47} + a_{5,1}e^{57} - a_{5,2}e^{67}, \\ de^2 &= a_{1,2}e^{17} + a_{1,1}e^{27} + a_{3,2}e^{37} + a_{3,1}e^{47} + a_{5,2}e^{57} + a_{5,1}e^{67}, \\ de^3 &= a_{1,3}e^{17} - a_{1,4}e^{27} + a_{3,3}e^{37} - a_{3,4}e^{47} + a_{5,3}e^{57} - a_{5,4}e^{67}, \\ de^4 &= a_{1,4}e^{17} + a_{1,3}e^{27} + a_{3,4}e^{37} + a_{3,3}e^{47} + a_{5,4}e^{57} + a_{5,3}e^{67}, \\ de^5 &= a_{1,5}e^{17} - a_{1,6}e^{27} + a_{3,5}e^{37} - a_{3,6}e^{47} + (-a_{1,1} - a_{3,3})e^{57} + (a_{1,2} + a_{3,4})e^{67}, \\ de^6 &= a_{1,6}e^{17} + a_{1,5}e^{27} + a_{3,6}e^{37} + a_{3,5}e^{47} + (-a_{1,2} - a_{3,4})e^{57} + (-a_{1,1} - a_{3,3})e^{67}, \\ de^7 &= 0. \end{aligned}$$

The condition of D being a derivation of \mathfrak{a} is equivalent to the vanishing of the differential operator when applied twice. From (10)

$$\begin{aligned} d^2e^1 &= 0, & d^2e^2 &= 0, & d^2e^3 &= 0, & d^2e^4 &= 0, \\ d^2e^5 &= 0, & d^2e^6 &= 0, & d^2e^7 &= 0, \end{aligned}$$

and therefore D is derivation of \mathfrak{a} for every $a_{1,1}, a_{1,2}, a_{1,3}, a_{1,4}, a_{1,5}, a_{1,6}, a_{3,1}, a_{3,2}, a_{3,3}, a_{3,4}, a_{3,5}, a_{3,6}, a_{5,1}, a_{5,2}, a_{5,3}, a_{5,4} \in \mathbb{R}$. Thus, the Lie algebra \mathfrak{g} which structure equations are described in (10) admits the closed G_2 -structure (2).

- $\mathfrak{h} = \mathfrak{e}(1, 1) \oplus \mathfrak{e}(1, 1)$

The structure equations of

$$\mathfrak{g} = (\mathfrak{e}(1, 1) \oplus \mathfrak{e}(1, 1)) \oplus_D \mathbb{R}e_7,$$

with D a derivation described as in (6) are

$$\begin{aligned}
de^1 &= a_{1,1}e^{17} - a_{1,2}e^{27} + a_{3,1}e^{37} - a_{3,2}e^{47} + a_{5,1}e^{57} - a_{5,2}e^{67}, \\
de^2 &= a_{1,2}e^{17} + a_{1,1}e^{27} + a_{3,2}e^{37} + a_{3,1}e^{47} + a_{5,2}e^{57} + a_{5,1}e^{67}, \\
de^3 &= -e^{14} + a_{1,3}e^{17} - a_{1,4}e^{27} + a_{3,3}e^{37} - a_{3,4}e^{47} + a_{5,3}e^{57} - a_{5,4}e^{67}, \\
de^4 &= -e^{13} + a_{1,4}e^{17} + a_{1,3}e^{27} + a_{3,4}e^{37} + a_{3,3}e^{47} + a_{5,4}e^{57} + a_{5,3}e^{67}, \\
de^5 &= e^{25} + a_{1,5}e^{17} - a_{1,6}e^{27} + a_{3,5}e^{37} - a_{3,6}e^{47} \\
&\quad + (-a_{1,1} - a_{3,3})e^{57} + (a_{1,2} + a_{3,4})e^{67}, \\
de^6 &= -e^{26} + a_{1,6}e^{17} + a_{1,5}e^{27} + a_{3,6}e^{37} + a_{3,5}e^{47} \\
&\quad + (-a_{1,2} - a_{3,4})e^{57} + (-a_{1,1} - a_{3,3})e^{67}, \\
de^7 &= 0.
\end{aligned}$$

The condition of D being a derivation of $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$ is equivalent to the vanishing of the differential operator when applied twice. Thus

$$\begin{aligned}
d^2e^1 &= a_{3,2}e^{137} - a_{3,1}e^{147} + a_{5,1}e^{257} + a_{5,2}e^{267}, \\
d^2e^2 &= -a_{3,1}e^{137} - a_{3,2}e^{147} + a_{5,2}e^{257} - a_{5,1}e^{267}, \\
d^2e^3 &= a_{1,3}e^{127} + 2a_{3,4}e^{137} + a_{1,1}e^{147} + a_{5,4}e^{157} + a_{5,3}e^{167} - a_{1,2}e^{247} + a_{5,3}e^{257} \\
&\quad + a_{5,4}e^{267} + a_{3,1}e^{347} - a_{5,1}e^{457} + a_{5,2}e^{467}, \\
d^2e^4 &= -a_{1,4}e^{127} + a_{1,1}e^{137} - 2a_{3,4}e^{147} + a_{5,3}e^{157} - a_{5,4}e^{167} - a_{1,2}e^{237} + a_{5,4}e^{257} \\
&\quad - a_{5,3}e^{267} + a_{3,2}e^{347} - a_{5,1}e^{357} + a_{5,2}e^{367}, \\
d^2e^5 &= a_{1,5}e^{127} + a_{3,6}e^{137} - a_{3,5}e^{147} - a_{1,2}e^{157} - a_{3,5}e^{237} + a_{3,6}e^{247} \\
&\quad - a_{1,1}e^{257} + (-2a_{1,2} - 2a_{3,4})e^{267} - a_{3,2}e^{357} - a_{3,1}e^{457} + a_{5,1}e^{567}, \\
d^2e^6 &= -a_{1,6}e^{127} - a_{3,5}e^{137} - a_{3,6}e^{147} + a_{1,2}e^{167} + a_{3,6}e^{237} + a_{3,5}e^{247} \\
&\quad + (-2a_{1,2} - 2a_{3,4})e^{257} + a_{1,1}e^{267} + a_{3,2}e^{367} + a_{3,1}e^{467} + a_{5,2}e^{567}, \\
d^2e^7 &= 0,
\end{aligned}$$

and after solving the system $d^2 = 0$ we conclude that the derivations $D \in \mathfrak{sl}(3, \mathbb{C})$ of $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$ are of the form

$$D = \begin{pmatrix} & & \\ & a_{33} & \\ & & a_{33} \\ & & & -a_{33} \\ & & & & -a_{33} \end{pmatrix}.$$

Therefore, the family of Lie algebras $\mathfrak{g} = (\mathfrak{e}(1, 1) \oplus \mathfrak{e}(1, 1)) \oplus_D \mathbb{R}e_7$ whose structure equations are

$$(\mathfrak{e}(1, 1) \oplus \mathfrak{e}(1, 1)) \oplus_D \mathbb{R}e_7 = (0, 0, -e^{14} + a_{3,3}e^{37}, -e^{13} + a_{3,3}e^{47}, e^{25} - a_{3,3}e^{57}, \\ -e^{26} - a_{3,3}e^{67})$$

admits the closed G_2 -structure defined by (2).

$$\bullet \quad \mathfrak{h} = \mathfrak{g}_{5,1} \oplus \mathbb{R}$$

The structure equations of

$$\mathfrak{g} = (\mathfrak{g}_{5,1} \oplus \mathbb{R}) \oplus_D \mathbb{R}e_7,$$

with D a derivation described as in (6) are

$$\begin{aligned} de^1 &= a_{1,1}e^{17} - a_{1,2}e^{27} + a_{3,1}e^{37} - a_{3,2}e^{47} + a_{5,1}e^{57} - a_{5,2}e^{67}, \\ de^2 &= a_{1,2}e^{17} + a_{1,1}e^{27} + a_{3,2}e^{37} + a_{3,1}e^{47} + a_{5,2}e^{57} + a_{5,1}e^{67}, \\ de^3 &= a_{1,3}e^{17} - a_{1,4}e^{27} + a_{3,3}e^{37} - a_{3,4}e^{47} + a_{5,3}e^{57} - a_{5,4}e^{67}, \\ de^4 &= e^{15} + a_{1,4}e^{17} + a_{1,3}e^{27} + a_{3,4}e^{37} + a_{3,3}e^{47} + a_{5,4}e^{57} + a_{5,3}e^{67}, \\ de^5 &= a_{1,5}e^{17} - a_{1,6}e^{27} + a_{3,5}e^{37} - a_{3,6}e^{47} + (-a_{1,1} - a_{3,3})e^{57} + (a_{1,2} + a_{3,4})e^{67}, \\ de^6 &= e^{13} + a_{1,6}e^{17} + a_{1,5}e^{27} + a_{3,6}e^{37} + a_{3,5}e^{47} + (-a_{1,2} - a_{3,4})e^{57} + (-a_{1,1} - a_{3,3})e^{67}, \\ de^7 &= 0. \end{aligned}$$

The condition of D being a derivation of $\mathfrak{g}_{5,1} \oplus \mathbb{R}$ is equivalent to the vanishing of the differential operator when applied twice. Thus

$$\begin{aligned} d^2e^1 &= -a_{5,2}e^{137} - a_{3,2}e^{157}, \\ d^2e^2 &= a_{5,1}e^{137} + a_{3,1}e^{157}, \\ d^2e^3 &= -a_{5,4}e^{137} - a_{3,4}e^{157}, \\ d^2e^4 &= a_{1,6}e^{127} + (a_{5,3} - a_{3,5})e^{137} + a_{3,6}e^{147} + 2a_{3,3}e^{157} + (-a_{1,2} - a_{3,4})e^{167} \\ &\quad + a_{1,2}e^{257} - a_{3,1}e^{357} + a_{3,2}e^{457} - a_{5,2}e^{567}, \\ d^2e^5 &= (a_{1,2} + a_{3,4})e^{137} - a_{3,6}e^{157}, \\ d^2e^6 &= a_{1,4}e^{127} + (-2a_{1,1} - 2a_{3,3})e^{137} + a_{3,4}e^{147} + (a_{3,5} - a_{5,3})e^{157} + a_{5,4}e^{167} \\ &\quad + a_{1,2}e^{237} - a_{3,2}e^{347} + a_{5,1}e^{357} - a_{5,2}e^{367}, \\ d^2e^7 &= 0, \end{aligned}$$

and after solving the system $d^2 = 0$ we conclude that the derivations $D \in \mathfrak{sl}(3, \mathbb{C})$ of $\mathfrak{g}_{5,1} \oplus \mathbb{R}$ are of the form

$$D = \left(\begin{array}{c|cc} & a_{1,3} & a_{1,5} \\ \hline & a_{1,3} & a_{1,5} \\ \hline & & a_{3,5} \\ & & a_{3,5} \\ \hline a_{3,5} & & \\ & a_{3,5} & \end{array} \right).$$

Therefore, the family of Lie algebras $\mathfrak{g} = (\mathfrak{g}_{5,1} \oplus \mathbb{R}) \oplus_D \mathbb{R}e_7$ whose structure equations are

$$(\mathfrak{g}_{5,1} \oplus \mathbb{R}) \oplus_D \mathbb{R}e_7 = (0, 0, a_{1,3}e^{17} + a_{3,5}e^{57}, e^{15} + a_{1,3}e^{27} + a_{3,5}e^{67}, a_{1,5}e^{17} + a_{3,5}e^{37}, e^{13} + a_{1,5}e^{27} + a_{3,5}e^{47}, 0)$$

are such that the G_2 -form (2) is closed.

- $\mathfrak{h} = \mathfrak{g}_{5,7}^{-1,-1,1} \oplus \mathbb{R}$

The structure equations of

$$\mathfrak{g} = (\mathfrak{g}_{5,7}^{-1,-1,1} \oplus \mathbb{R}) \oplus_D \mathbb{R}e_7,$$

with D a derivation described as in (6) are

$$\begin{aligned} de^1 &= -e^{15} + a_{1,1}e^{17} - a_{1,2}e^{27} + a_{3,1}e^{37} - a_{3,2}e^{47} + a_{5,1}e^{57} - a_{5,2}e^{67}, \\ de^2 &= e^{25} + a_{1,2}e^{17} + a_{1,1}e^{27} + a_{3,2}e^{37} + a_{3,1}e^{47} + a_{5,2}e^{57} + a_{5,1}e^{67}, \\ de^3 &= -e^{35} + a_{1,3}e^{17} - a_{1,4}e^{27} + a_{3,3}e^{37} - a_{3,4}e^{47} + a_{5,3}e^{57} - a_{5,4}e^{67}, \\ de^4 &= e^{45} + a_{1,4}e^{17} + a_{1,3}e^{27} + a_{3,4}e^{37} + a_{3,3}e^{47} + a_{5,4}e^{57} + a_{5,3}e^{67}, \\ de^5 &= a_{1,5}e^{17} - a_{1,6}e^{27} + a_{3,5}e^{37} - a_{3,6}e^{47} + (-a_{1,1} - a_{3,3})e^{57} + (a_{1,2} + a_{3,4})e^{67}, \\ de^6 &= a_{1,6}e^{17} + a_{1,5}e^{27} + a_{3,6}e^{37} + a_{3,5}e^{47} + (-a_{1,2} - a_{3,4})e^{57} + (-a_{1,1} - a_{3,3})e^{67}, \\ de^7 &= 0. \end{aligned}$$

Thus after imposing $d^2 = 0$ we conclude that the real representations of the derivations $D \in \mathfrak{sl}(3, \mathbb{C})$ of $\mathfrak{g}_{5,7}^{-1,-1,1} \oplus \mathbb{R}$ are

$$D = \left(\begin{array}{c|cc} a_{1,1} & a_{1,3} & \\ \hline a_{1,1} & a_{1,3} & \\ \hline a_{3,1} & -a_{1,1} & \\ a_{3,1} & -a_{1,1} & \\ \hline & & \end{array} \right).$$

Hence, the family of Lie algebras $\mathfrak{g} = (\mathfrak{g}_{5,7}^{-1,-1,1} \oplus \mathbb{R}) \oplus_D \mathbb{R}e_7$ whose structure equations are

$$(\mathfrak{g}_{5,7}^{-1,-1,1} \oplus \mathbb{R}) \oplus_D \mathbb{R}e_7 = (-e^{15} + a_{1,3}e^{17} - a_{1,1}e^{37}, e^{25} + a_{1,1}e^{17} + a_{3,1}e^{37}, \\ -e^{35} + a_{1,3}e^{17} - a_{1,1}e^{37}, e^{45} + a_{1,3}e^{27} - a_{1,1}e^{47}, 0, 0, 0)$$

is such that the G_2 -form (2) is closed for all $a_{1,1}, a_{1,3}$ and $a_{3,1}$ real numbers.

- $\mathfrak{h} = \mathfrak{g}_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ with $\alpha \geq 0$

The structure equations of

$$\mathfrak{g} = (\mathfrak{g}_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}) \oplus_D \mathbb{R}e_7,$$

with D as in (6) are

$$\begin{aligned} de^1 &= \alpha e^{15} + e^{35} + a_{1,1}e^{17} - a_{1,2}e^{27} + a_{3,1}e^{37} - a_{3,2}e^{47} + a_{5,1}e^{57} - a_{5,2}e^{67}, \\ de^2 &= -\alpha e^{25} + e^{45} + a_{1,2}e^{17} + a_{1,1}e^{27} + a_{3,2}e^{37} + a_{3,1}e^{47} + a_{5,2}e^{57} + a_{5,1}e^{67}, \\ de^3 &= -e^{15} + \alpha e^{35} + a_{1,3}e^{17} - a_{1,4}e^{27} + a_{3,3}e^{37} - a_{3,4}e^{47} + a_{5,3}e^{57} - a_{5,4}e^{67}, \\ de^4 &= -e^{25} - \alpha e^{45} + a_{1,4}e^{17} + a_{1,3}e^{27} + a_{3,4}e^{37} + a_{3,3}e^{47} + a_{5,4}e^{57} + a_{5,3}e^{67}, \\ de^5 &= a_{1,5}e^{17} - a_{1,6}e^{27} + a_{3,5}e^{37} - a_{3,6}e^{47} + (-a_{1,1} - a_{3,3})e^{57} + (a_{1,2} + a_{3,4})e^{67}, \\ de^6 &= a_{1,6}e^{17} + a_{1,5}e^{27} + a_{3,6}e^{37} + a_{3,5}e^{47} + (-a_{1,2} - a_{3,4})e^{57} + (-a_{1,1} - a_{3,3})e^{67}, \\ de^7 &= 0. \end{aligned}$$

As before we impose the condition $d^2 = 0$, obtaining that the derivations $D \in \mathfrak{sl}(3, \mathbb{C})$ of $\mathfrak{g}_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ are

$$D = \left(\begin{array}{c|c|c} & a_{1,3} & \\ \hline -a_{1,3} & & a_{1,3} \\ \hline & -a_{1,3} & \\ \hline & & \end{array} \right).$$

Therefore, the family of Lie algebras $\mathfrak{g} = (\mathfrak{g}_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}) \oplus_D \mathbb{R}e_7$ with $\alpha \geq 0$, whose structure equations are

$$(\mathfrak{g}_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}) \oplus_D \mathbb{R}e_7 = (\alpha e^{15} + e^{35} - a_{1,3}e^{37}, -\alpha e^{25} + e^{45} - a_{1,3}e^{47}, \\ -e^{15} + \alpha e^{35} + a_{1,3}e^{17}, -e^{25} - \alpha e^{45} + a_{1,3}e^{27}, 0, 0, 0)$$

satisfies that the G_2 -form described by (2) is closed for all $\alpha \geq 0$ and $a_{1,3}$ real number.

- $\mathfrak{h} = \mathfrak{g}_{6,N3}$

The structure equations of

$$\mathfrak{g} = \mathfrak{g}_{6,N3} \oplus_D \mathbb{R}e_7,$$

with D the real representation of certain $A \in \mathfrak{sl}(3, \mathbb{C})$ are

$$\begin{aligned}
de^1 &= a_{1,1}e^{17} - a_{1,2}e^{27} + a_{3,1}e^{37} - a_{3,2}e^{47} + a_{5,1}e^{57} - a_{5,2}e^{67}, \\
de^2 &= e^{35} + a_{1,2}e^{17} + a_{1,1}e^{27} + a_{3,2}e^{37} + a_{3,1}e^{47} + a_{5,2}e^{57} + a_{5,1}e^{67}, \\
de^3 &= a_{1,3}e^{17} - a_{1,4}e^{27} + a_{3,3}e^{37} - a_{3,4}e^{47} + a_{5,3}e^{57} - a_{5,4}e^{67}, \\
de^4 &= 2e^{15} + a_{1,4}e^{17} + a_{1,3}e^{27} + a_{3,4}e^{37} + a_{3,3}e^{47} + a_{5,4}e^{57} + a_{5,3}e^{67}, \\
de^5 &= a_{1,5}e^{17} - a_{1,6}e^{27} + a_{3,5}e^{37} - a_{3,6}e^{47} + (-a_{1,1} - a_{3,3})e^{57} + (a_{1,2} + a_{3,4})e^{67}, \\
de^6 &= e^{13} + a_{1,6}e^{17} + a_{1,5}e^{27} + a_{3,6}e^{37} + a_{3,5}e^{47} + (-a_{1,2} - a_{3,4})e^{57} \\
&\quad + (-a_{1,1} - a_{3,3})e^{67}, \\
de^7 &= 0.
\end{aligned}$$

Proceeding as before we obtain that the derivation D is described by

$$D = \left(\begin{array}{c|c|c} & a_{1,3} & a_{1,5} \\ \hline & a_{1,3} & a_{1,5} \\ \hline \frac{a_{1,3}}{2} & & a_{3,5} \\ & \frac{a_{1,3}}{2} & a_{3,5} \\ \hline -a_{1,5} & 2a_{3,5} & \\ & -a_{1,5} & 2a_{3,5} \end{array} \right).$$

Thus, the family of Lie algebras $\mathfrak{g} = \mathfrak{g}_{6,N3} \oplus_D \mathbb{R}e_7$ has structure equations

$$\begin{aligned}
\mathfrak{g}_{6,N3} \oplus_D \mathbb{R}e_7 = & \left(\frac{a_{1,3}}{2}e^{37} - a_{1,5}e^{57}, e^{35} + \frac{a_{1,3}}{2}e^{47} - a_{1,5}e^{67}, a_{1,3}e^{17} + 2a_{3,5}e^{57}, \right. \\
& \left. a_{1,3}e^{27} + 2a_{3,5}e^{67}, a_{1,5}e^{17} + a_{3,5}e^{37}, e^{13} + a_{1,5}e^{27} + a_{3,5}e^{47}, 0 \right).
\end{aligned}$$

Hence, the G_2 form (2) is closed for all $a_{1,3}, a_{1,5}$ and $a_{3,5}$ real numbers.

- $\mathfrak{g} = \mathfrak{g}_{6,38}^0$

The structure equations of

$$\mathfrak{g} = \mathfrak{g}_{6,38}^0 \oplus_D \mathbb{R}e_7,$$

with D as in (6) are

$$\begin{aligned}
de^1 &= 2e^{36} + a_{1,1}e^{17} - a_{1,2}e^{27} + a_{3,1}e^{37} - a_{3,2}e^{47} + a_{5,1}e^{57} - a_{5,2}e^{67}, \\
de^2 &= a_{1,2}e^{17} + a_{1,1}e^{27} + a_{3,2}e^{37} + a_{3,1}e^{47} + a_{5,2}e^{57} + a_{5,1}e^{67}, \\
de^3 &= -e^{26} + a_{1,3}e^{17} - a_{1,4}e^{27} + a_{3,3}e^{37} - a_{3,4}e^{47} + a_{5,3}e^{57} - a_{5,4}e^{67}, \\
de^4 &= -e^{26} + e^{25} + a_{1,4}e^{17} + a_{1,3}e^{27} + a_{3,4}e^{37} + a_{3,3}e^{47} + a_{5,4}e^{57} + a_{5,3}e^{67}, \\
de^5 &= -e^{23} - e^{24} + a_{1,5}e^{17} - a_{1,6}e^{27} + a_{3,5}e^{37} - a_{3,6}e^{47} + (-a_{1,1} - a_{3,3})e^{57} \\
&\quad + (a_{1,2} + a_{3,4})e^{67}, \\
de^6 &= e^{23} + a_{1,6}e^{17} + a_{1,5}e^{27} + a_{3,6}e^{37} + a_{3,5}e^{47} + (-a_{1,2} - a_{3,4})e^{57} \\
&\quad + (-a_{1,1} - a_{3,3})e^{67}, \\
de^7 &= 0.
\end{aligned}$$

Imposing $d^2 = 0$ we obtain that the unique derivation D of $\mathfrak{g}_{6,38}^0$ of the form (6) is given by the null matrix. Thus, \mathfrak{g} is direct sum, that is

$$\mathfrak{g}_{6,38}^0 \oplus \mathbb{R}e_7 = (2e^{36}, 0, -e^{26}, -e^{26} + e^{25}, -e^{23} - e^{24}, e^{23}, 0).$$

It is clear that for this Lie algebra the G_2 form (2) is closed.

- $\mathfrak{h} = \mathfrak{g}_{6,54}^{0,-1}$

Exactly as before we have that

$$\mathfrak{g} = \mathfrak{g}_{6,54}^{0,-1} \oplus_D \mathbb{R}e_7,$$

with D as in (6) is exactly a direct sum, that is

$$\mathfrak{g}_{6,54}^{0,-1} \oplus \mathbb{R}e_7 = (e^{16} + e^{45}, -e^{26}, -e^{36} + e^{25}, e^{46}, 0, 0, 0),$$

and therefore the G_2 form (2) is closed for this algebra.

- $\mathfrak{h} = \mathfrak{g}_{6,118}^{0,-1,-1}$

The structure equations of

$$\mathfrak{g} = \mathfrak{g}_{6,118}^{0,-1,-1} \oplus_D \mathbb{R}e_7,$$

with D the real representation of certain 3×3 complex matrix without trace are

$$\begin{aligned} de^1 &= -e^{15} + e^{36} + a_{1,1}e^{17} - a_{1,2}e^{27} + a_{3,1}e^{37} - a_{3,2}e^{47} + a_{5,1}e^{57} - a_{5,2}e^{67}, \\ de^2 &= e^{46} + e^{25} + a_{1,2}e^{17} + a_{1,1}e^{27} + a_{3,2}e^{37} + a_{3,1}e^{47} + a_{5,2}e^{57} + a_{5,1}e^{67}, \\ de^3 &= -e^{16} - e^{35} + a_{1,3}e^{17} - a_{1,4}e^{27} + a_{3,3}e^{37} - a_{3,4}e^{47} + a_{5,3}e^{57} - a_{5,4}e^{67}, \\ de^4 &= e^{45} - e^{26} + a_{1,4}e^{17} + a_{1,3}e^{27} + a_{3,4}e^{37} + a_{3,3}e^{47} + a_{5,4}e^{57} + a_{5,3}e^{67}, \\ de^5 &= a_{1,5}e^{17} - a_{1,6}e^{27} + a_{3,5}e^{37} - a_{3,6}e^{47} + (-a_{1,1} - a_{3,3})e^{57} \\ &\quad + (a_{1,2} + a_{3,4})e^{67}, \\ de^6 &= a_{1,6}e^{17} + a_{1,5}e^{27} + a_{3,6}e^{37} + a_{3,5}e^{47} + (-a_{1,2} - a_{3,4})e^{57} \\ &\quad + (-a_{1,1} - a_{3,3})e^{67}, \\ de^7 &= 0. \end{aligned}$$

Solving the equation obtained from the fact that D has to be a derivation of $\mathfrak{g}_{6,118}^{0,-1,-1}$ we have that $D \in \mathfrak{sl}(3, \mathbb{C})$ is such that

$$D = \left(\begin{array}{c|c|c} & a_{1,3} & \\ \hline & a_{1,3} & \\ \hline -a_{1,3} & & \\ & -a_{1,3} & \\ \hline & & \end{array} \right).$$

Thus, from Proposition 1.1 the family of Lie algebras with structure equations

$$\mathfrak{g}_{6,118}^{0,-1,-1} \oplus_D \mathbb{R}e_7 = (-e^{15} + e^{36} - a_{1,3}e^{37}, e^{46} + e^{25} - a_{1,3}e^{47}, -e^{16} - e^{35} + a_{1,3}e^{17}, \\ e^{45} - e^{26} - a_{1,3}e^{27}, 0, 0, 0)$$

is such that the G_2 form (2) is closed for all $a_{1,3}$ real number.

- $\mathfrak{h} = A_{6,13}^{-\frac{2}{3}, \frac{1}{3}, -1}$

With the same procedure as for the previous Lie algebras we have that a derivation D of the Lie algebra $A_{6,13}^{-\frac{2}{3}, \frac{1}{3}, -1}$ being the real representation of a 3×3 complex matrix without trace has to be of the form

$$D = \left(\begin{array}{c|c|c} a_{1,1} & & \\ \hline & a_{1,1} & \\ \hline & & \\ \hline & & -a_{1,1} \\ & & -a_{1,1} \end{array} \right).$$

Therefore, from Proposition 1.1 the family of Lie algebras with structure equations

$$A_{6,13}^{-\frac{2}{3}, \frac{1}{3}, -1} \oplus_D \mathbb{R}e_7 = \left(-\frac{1}{4}e^{14} - e^{23} + a_{1,1}e^{17}, \frac{1}{4}e^{24} + a_{1,1}e^{27}, -\frac{1}{2}e^{34}, 0, \right. \\ \left. -\frac{3}{4}e^{45} - a_{1,1}e^{57}, \frac{3}{4}e^{46} - a_{1,1}e^{67}, 0 \right)$$

is such that the G_2 form (2) is closed for all $a_{1,1}$ real number.

- $\mathfrak{h} = A_{6,54}^{2,1}$

For this Lie algebra can be checked that the derivations D of $A_{6,54}^{2,1}$ of the form (6) are such that

$$D = \left(\begin{array}{c|c|c} a_{1,1} & a_{1,3} & \\ \hline & a_{1,3} & \\ \hline a_{3,1} & -a_{1,1} & \\ & -a_{1,1} & \\ \hline & & \end{array} \right).$$

Henceforth, the family of Lie algebras with structure equations

$$A_{6,54}^{2,1} \oplus_D \mathbb{R}e_7 = \left(-\frac{1}{2}e^{15} + a_{1,1}e^{17} + a_{3,1}e^{37}, \frac{1}{2}e^{25} + e^{16} + a_{1,1}e^{27} + a_{3,1}e^{47}, \right. \\ \left. -\frac{1}{2}e^{35} + a_{1,3}e^{17} - a_{1,1}e^{37}, -\frac{1}{2}e^{45} + e^{36} + a_{1,3}e^{27} - a_{1,1}e^{47}, 0, e^{56}, 0 \right)$$

is such that the G_2 form (2) is closed for all $a_{1,1}$, $a_{1,3}$ and $a_{3,1}$ real numbers.

- $\mathfrak{h} = A_{6,70}^{\alpha, \frac{\alpha}{2}} (\alpha \neq 0)$

For this Lie algebra the derivations of the form (6) are such that

$$D = \begin{pmatrix} & & a_{1,3} \\ & a_{1,3} & \\ -a_{1,3} & & \\ & -a_{1,3} & \\ & & \end{pmatrix}.$$

Therefore, the family of Lie algebras with structure equations

$$A_{6,70}^{\alpha, \frac{\alpha}{2}} \oplus_D \mathbb{R}e_7 = \left(-\frac{1}{2}e^{15} + \frac{1}{\alpha}e^{35} + e^{26} - a_{1,3}e^{37}, \frac{1}{2}e^{25} + \frac{1}{\alpha}e^{45} - a_{1,3}e^{47}, \right. \\ \left. -\frac{1}{\alpha}e^{15} - \frac{1}{2}e^{35} + e^{46} + a_{1,3}e^{17}, -\frac{1}{\alpha}e^{25} + \frac{1}{2}e^{45} + a_{1,3}e^{27}, 0, e^{56}, 0 \right)$$

is such that the G_2 form (2) is closed for any $a_{1,3} \in \mathbb{R}$.

- $\mathfrak{h} = A_{6,71}^{-\frac{3}{2}}$

The structure equations of

$$\mathfrak{g} = A_{6,71}^{-\frac{3}{2}} \oplus_D \mathbb{R}e_7,$$

with D as in (6) are

$$\begin{aligned} de^1 &= -\frac{3}{4}e^{16} + a_{1,1}e^{17} - a_{1,2}e^{27} + a_{3,1}e^{37} - a_{3,2}e^{47} + a_{5,1}e^{57} - a_{5,2}e^{67}, \\ de^2 &= \frac{3}{4}e^{26} + e^{35} + a_{1,2}e^{17} + a_{1,1}e^{27} + a_{3,2}e^{37} + a_{3,1}e^{47} + a_{5,2}e^{57} + a_{5,1}e^{67}, \\ de^3 &= \frac{1}{4}e^{36} + e^{45} + a_{1,3}e^{17} - a_{1,4}e^{27} + a_{3,3}e^{37} - a_{3,4}e^{47} + a_{5,3}e^{57} - a_{5,4}e^{67}, \\ de^4 &= -\frac{1}{4}e^{46} + e^{15} + a_{1,4}e^{17} + a_{1,3}e^{27} + a_{3,4}e^{37} + a_{3,3}e^{47} + a_{5,4}e^{57} + a_{5,3}e^{67}, \\ de^5 &= \frac{1}{2}e^{56} + a_{1,5}e^{17} - a_{1,6}e^{27} + a_{3,5}e^{37} - a_{3,6}e^{47} + (-a_{1,1} - a_{3,3})e^{57} \\ &\quad + (a_{1,2} + a_{3,4})e^{67}, \\ de^6 &= a_{1,6}e^{17} + a_{1,5}e^{27} + a_{3,6}e^{37} + a_{3,5}e^{47} + (-a_{1,2} - a_{3,4})e^{57} \\ &\quad + (-a_{1,1} - a_{3,3})e^{67}, \\ de^7 &= 0. \end{aligned}$$

The condition $d^2 = 0$ is satisfied if and only if all the parameters vanish. Thus \mathfrak{g} is the direct sum

$$A_{6,71}^{-\frac{3}{2}} \oplus \mathbb{R}e_7 = \left(-\frac{3}{4}e^{16}, \frac{3}{4}e^{26} + e^{35}, \frac{1}{4}e^{36} + e^{45}, -\frac{1}{4}e^{46} + e^{15}, \frac{1}{2}e^{56}, 0, 0 \right),$$

and the G_2 form (2) is closed.

- $\mathfrak{h} = N_{6,13}^{0,-2,0,-2}$

The structure equations of

$$N_{6,13}^{0,-2,0,-2} \oplus_D \mathbb{R}e_7,$$

with D as in (6) are

$$\begin{aligned} de^1 &= -2 \cdot 3^{-1/6} e^{16} + a_{1,1} e^{17} - a_{1,2} e^{27} + a_{3,1} e^{37} - a_{3,2} e^{47} + a_{5,1} e^{57} - a_{5,2} e^{67}, \\ de^2 &= 2 \cdot 3^{-1/6} e^{26} + e^{35} + a_{1,2} e^{17} + a_{1,1} e^{27} + a_{3,2} e^{37} + a_{3,1} e^{47} + a_{5,2} e^{57} + a_{5,1} e^{67}, \\ de^3 &= 3^{-1/6} e^{36} + 3^{5/6} e^{45} + a_{1,3} e^{17} - a_{1,4} e^{27} + a_{3,3} e^{37} - a_{3,4} e^{47} + a_{5,3} e^{57} - a_{5,4} e^{67}, \\ de^4 &= a_{1,4} e^{17} + a_{1,3} e^{27} + a_{3,4} e^{37} + a_{3,3} e^{47} + a_{5,4} e^{57} + a_{5,3} e^{67}, \\ de^5 &= 3^{-1/6} e^{34} + 3^{-1/6} e^{56} + a_{1,5} e^{17} - a_{1,6} e^{27} + a_{3,5} e^{37} - a_{3,6} e^{47} + (-a_{1,1} - a_{3,3}) e^{57} \\ &\quad + (a_{1,2} + a_{3,4}) e^{67}, \\ de^6 &= a_{1,6} e^{17} + a_{1,5} e^{27} + a_{3,6} e^{37} + a_{3,5} e^{47} + (-a_{1,2} - a_{3,4}) e^{57} \\ &\quad + (-a_{1,1} - a_{3,3}) e^{67}, \\ de^7 &= 0. \end{aligned}$$

Thus, imposing the condition $d^2 = 0$ we obtain that the unique derivation of $N_{6,13}^{0,-2,0,-2}$ of the form (6) is given by the null matrix. Hence, \mathfrak{g} is the direct sum

$$\begin{aligned} N_{6,13}^{0,-2,0,-2} \oplus \mathbb{R}e_7 &= (-2 \cdot 3^{-1/6} e^{16}, 2 \cdot 3^{-1/6} e^{26}, 3^{-1/6} e^{36} + 3^{5/6} e^{45}, 0, \\ &\quad 3^{-1/6} e^{34} + 3^{-1/6} e^{56}, 0, 0). \end{aligned}$$

Therefore, the G_2 form (2) is closed.

□

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4. APPENDIX

TABLE 1. Lie algebras endowed with a closed G_2 -structure obtained in Proposition 3.1

| \mathfrak{g} | Structure equations |
|---|---|
| $\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1) \oplus_D \mathbb{R}e_7$ | $(0, 0, -e^{14} + a_{3,3}e^{37}, -e^{13} + a_{3,3}e^{47}, e^{25} - a_{3,3}e^{57}, -e^{26} - a_{3,3}e^{67})$ |
| $(\mathfrak{g}_{5,1} \oplus \mathbb{R}) \oplus_D \mathbb{R}e_7$ | $(0, 0, a_{1,3}e^{17} + a_{3,5}e^{57}, e^{15} + a_{1,3}e^{27} + a_{3,5}e^{67}, a_{1,5}e^{17} + a_{3,5}e^{37}, e^{13} + a_{1,5}e^{27} + a_{3,5}e^{47}, 0)$ |
| $(\mathfrak{g}_{5,7}^{-1,-1,1} \oplus \mathbb{R}) \oplus_D \mathbb{R}e_7$ | $(-e^{15} + a_{1,3}e^{17} + a_{1,1}e^{37}, e^{25} + a_{1,1}e^{17} + a_{3,1}e^{37}, -e^{35} + a_{1,3}e^{17} + a_{1,1}e^{37}, e^{45} + a_{1,3}e^{27} - a_{1,1}e^{47}, 0, 0, 0)$ |
| $(\mathfrak{g}_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}) \oplus_D \mathbb{R}e_7$ | $(\alpha e^{15} + e^{35} - a_{1,3}e^{37}, -\alpha e^{25} + e^{45} - a_{1,3}e^{47}, -e^{15} + \alpha e^{35} + a_{1,3}e^{17}, -e^{25} - \alpha e^{45} + a_{1,3}e^{27}, 0, 0, 0)$ |
| $\mathfrak{g}_{6,N3} \oplus_D \mathbb{R}e_7$ | $(\frac{a_{1,3}}{2}e^{37} - a_{1,5}e^{57}, e^{35} + \frac{a_{1,3}}{2}e^{47} - a_{1,5}e^{67}, a_{1,3}e^{17} + 2a_{3,5}e^{57}, a_{1,3}e^{27} + 2a_{3,5}e^{67}, a_{1,5}e^{17} + a_{3,5}e^{37}, e^{13} + a_{1,5}e^{27} + a_{3,5}e^{47}, 0)$ |
| $\mathfrak{g}_{6,38}^0 \oplus \mathbb{R}e_7$ | $(2e^{36}, 0, -e^{26}, -e^{26} + e^{25}, -e^{23} - e^{24}, e^{23}, 0)$ |
| $\mathfrak{g}_{6,54}^{0,-1} \oplus \mathbb{R}e_7$ | $(e^{16} + e^{45}, -e^{26}, -e^{36} + e^{25}, e^{46}, 0, 0, 0)$ |
| $\mathfrak{g}_{6,118}^{0,-1,-1} \oplus_D \mathbb{R}e_7$ | $(-e^{15} + e^{36} - a_{1,3}e^{37}, e^{46} + e^{25} - a_{1,3}e^{47}, -e^{16} - e^{35} + a_{1,3}e^{17}, e^{45} - e^{26} - a_{1,3}e^{27}, 0, 0, 0)$ |
| $A_{6,13}^{-\frac{2}{3}, \frac{1}{3}, -1} \oplus_D \mathbb{R}e_7$ | $(-\frac{1}{4}e^{14} - e^{23} + a_{1,1}e^{17}, \frac{1}{4}e^{24} + a_{1,1}e^{27}, -\frac{1}{2}e^{34}, 0, -\frac{3}{4}e^{45} - a_{1,1}e^{57}, \frac{3}{4}e^{46} - a_{1,1}e^{67}, 0)$ |
| $A_{6,54}^{2,1} \oplus_D \mathbb{R}e_7$ | $(-\frac{1}{2}e^{15} + a_{1,1}e^{17} + a_{3,1}e^{37}, \frac{1}{2}e^{25} + e^{16} + a_{1,1}e^{27} + a_{3,1}e^{47}, -\frac{1}{2}e^{35} + a_{1,3}e^{17} - a_{1,1}e^{37}, -\frac{1}{2}e^{45} + e^{36} + a_{1,3}e^{27} - a_{1,1}e^{47}, 0, e^{56}, 0)$ |
| $A_{6,70}^{\alpha, \frac{\alpha}{2}} \oplus_D \mathbb{R}e_7$ | $(-\frac{1}{2}e^{15} + \frac{1}{\alpha}e^{35} + e^{26} - a_{1,3}e^{37}, \frac{1}{2}e^{25} + \frac{1}{\alpha}e^{45} - a_{1,3}e^{47}, -\frac{1}{\alpha}e^{15} - \frac{1}{2}e^{35} + e^{46} + a_{1,3}e^{17}, -\frac{1}{\alpha}e^{25} + \frac{1}{2}e^{45} + a_{1,3}e^{27}, 0, e^{56}, 0)$ |
| $A_{6,71}^{-\frac{3}{2}} \oplus \mathbb{R}e_7$ | $(-\frac{3}{4}e^{16}, \frac{3}{4}e^{26} + e^{35}, \frac{1}{4}e^{36} + e^{45}, -\frac{1}{4}e^{46} + e^{15}, \frac{1}{2}e^{56}, 0, 0)$ |
| $N_{6,13}^{0,-2,0,-2} \oplus \mathbb{R}e_7$ | $(-2 \cdot 3^{-1/6}e^{16}, 2 \cdot 3^{-1/6}e^{26}, 3^{-1/6}e^{36} + 3^{5/6}e^{45}, 0, 3^{-1/6}e^{34} + 3^{-1/6}e^{56}, 0, 0)$ |

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